A solution for tubular structures of all kinds through implementation of an inverse solution

Abstract

Numerous formulations of constitutive models for arteries have been proposed in the literature and overviews are provided by, e.g., Humphrey [6] and Holzapfel et al. [1]. For evaluation of several of the most commonly used arterial wall models we refer to, e.g., Humphrey [5], Holzapfel et al. [1] and Horgan and Saccomandi [4], among others. [7] Traditional constitutive frameworks for high-strain materials are ill-suited to solve extension and inflation, one of the simplest problems involving tubes, or more complicated problems. Moreover, it is experimentally necessary to minimize the covariance amongst constitutive response functions [9]. The author's claim that the covariance in terms is why models such as Chuong, C. J., Fung, Y. C [8] is what cause the model to be a non-solution in the strain energy domain. The solution presented is third order and practical when one beings to examine the physiology and mechanical behavior of carotid arteries in all species with a four chambered heart. The final solution takes into account a previous conjecture of having a minimal energy conformation in the $(\alpha - c)^2$ granting insight into the mechanical properties of carotid arteries. The purpose of this paper is to present the necessity of the usage of a constitutive frame work that permits invertitibility. Further, to the authors' knowledge all frameworks that permit invertibiltiy minimize covariance, which the authors' claim is necessary to obtain a solution for the proposed problem.

Key words: strain energy, solution, constitutive behavior, finite elasticity, biomechanics, mechanics

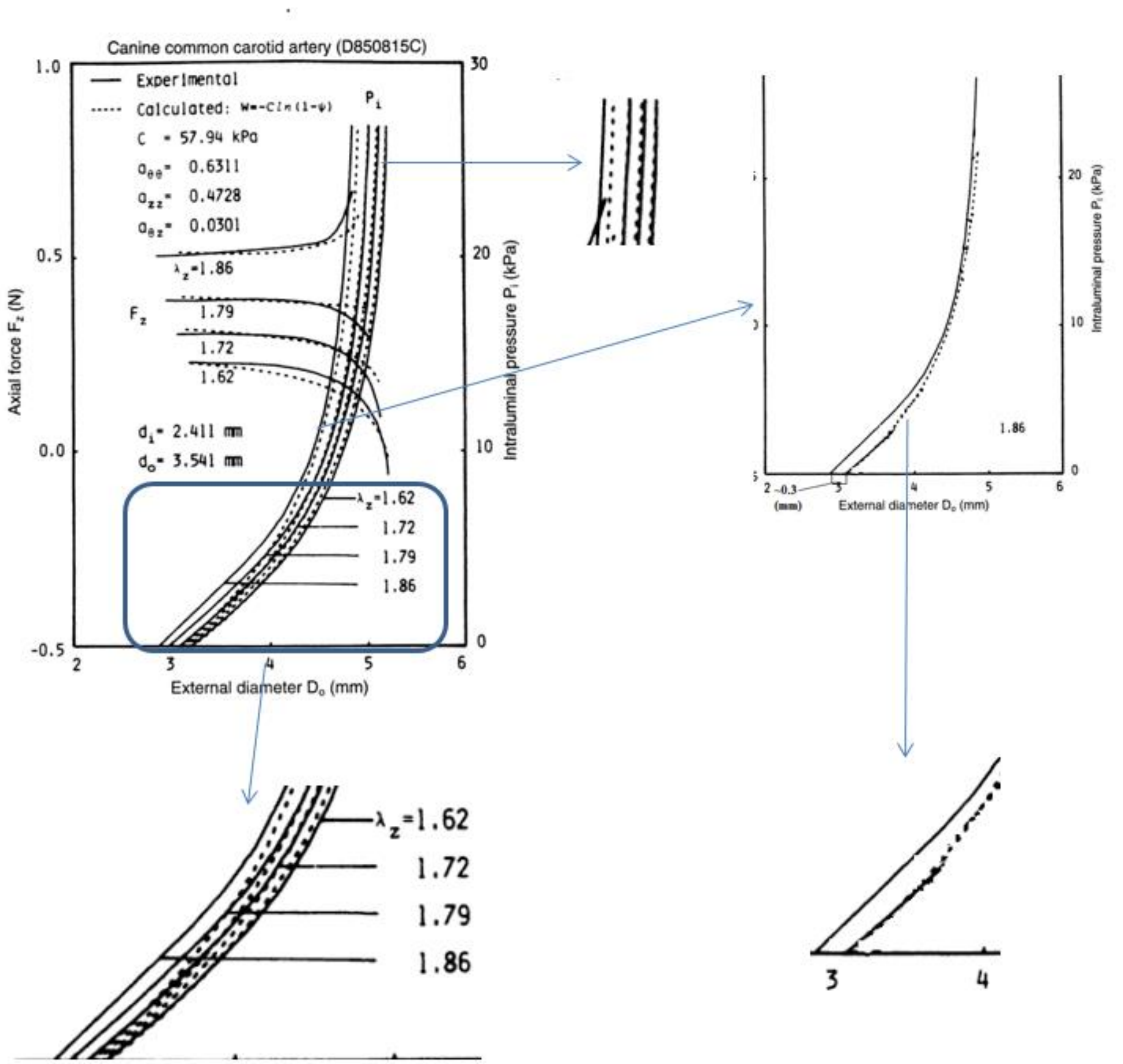
Introduction

Improvements in knowledge of mechanical behavior of certain tissues has long been sought after. An understating of how certain tissues principally behave will grant avenues to greater understanding of the human body. Much work has been successfully performed in biomechanics in an attempt to gain a better understanding of the basic mechanical properties. It is of the authors' opinion that the first great leap from models that granted little understanding to truly gaining insight into how soft tissues behave was Dr. Fung's interpretation of his model from 1983. Dr. Fung sought to show that the use of *average* circumferential stress in modeling characteristic strains was ill-conceived. Since then his findings have been verified and the general consensus is that circumferential stress in the vessel wall is non-uniform thus, showing that the previous assumption that usage of average circumferential stress in relating characteristics strains to this value were ill-conceived. [8]

Since Dr. Fung's paper much work has been done on optimizing his original strain energy function. The need for optimizing the original strain energy arises from inaccuracies that can arise when implementing it. The cause of these in accuracies has been well documented and investigated by Walton Et. Al. [10]

To the author's knowledge all models that have been previously conceived and published would fall into not being strongly elliptic since they ascertain and mimic the characteristics of Fung 1983 which are causal to lacking strong ellipticity. Beyond meeting the criterion for strong ellipticity we see a failure to match experimental data. For Example, Takamizawa 1987 proposes a logarithmic type of strain energy density function was used to describe the wall properties.

Seemingly to the authors' the model is proposed as an improvement of previous models, one which grants greater accuracy and more insight into the mechanical behavior and cardiovascular physiology. Yet, the model is built upon the inverse function of what Dr. Fung has used which one could reasonably conclude would yield quasi Fung results. Being that Fung has been widely accepted as the model for soft tissue behavior we can reasonably foresee results that would mimic previous attempts. Takamizawa's results are shown below:



To the author's opinion the following model: 1) Only fits once the exponential regime is reached 2) Fits meagerly at best in the exponential regime, meaning fit in shape (exponential shape) but not in magnitude (refer to the isolated the lambda=1.86 curve; at top right). 3) Is a non-solution for small strain 4) Lacks linearity for data at intramural pressures between 0 and 7 Kpa.

For those who are unfamiliar with Implementation of Strain Energy See Criscione 2004 [9] for a detailed explanation on implementation of constitutive equations.

The authors' last introductory question is how does a "Fung-like" strain energy density function fair at modeling strain energy? To the author's knowledge prior to this paper strain energy for tubular structures has only been speculated and never truly known. In other words, when a model was found to reasonably fit in a few regimes (load, pressure, torsion) with a set of parameters, those parameters were then used with the strain energy and this was defined as "strain energy". But, with the understanding that the constitutive frameworks that have previously been used produce an over determined solution. Thus, the statistical likely hood of the garnered strain energy reasonably modeling the inherent and unique strain energy of a system is statitilsically unlikely.

Methods

To arrive at the inverted strain energy function the authors used balance equations that are outlined that are simplified when using a Criscione's framework [9]. Then approaching the problem as simple extension and inflation of a tube, we can arrive at the strain energy function from our measurements of external diameter and axial length.

The models were fit using a least squares fit. This was done when fitting strain energy to subsequently executing the forward problem resulting in pressure and load. As well as fitting response data (Pressure, Load) and integrating to result in strain energy with the parameters. In the case of Fung's 1983 Model, a direct substitution of frame work was done for a clearer comparison to amongst the models.

Kinematics and Balance Equations for Straight Tube Model of Carotid Artery

For our analysis we consider a straight tube structure that under goes only deformation in the radial and axial direction. This greatly simplifies the analysis and thus we only consider deformations of extension and inflation. Discussion on the limitations of the model by not considering deformations of the telescoping and torsion will be left for the discussion portion.

Considering a structure that is carotid tubular, i.e. a structure that is long, has a lumen and a thick wall (Thick wall is where there is significant difference between interior and external diameter). Conceptually, let us think of this thick-walled structure as composed of many (perhaps infinitely many) thin-walled tubes or shells with a thickness variation similar to that of the thick-walled structure. In other words, if the tube thickness is uniform, then let the thickness of each shell be uniform; or if the tube is thicker in some regions, then let the shells be thicker in those regions. At a particular point of interest *Pt* in the reference configuration, let R be the "radial direction" or more precisely the unit vector that is normal to the shell and pointing away from the lumen. If the shell is not axis-symmetric, then R may not necessarily be the radial direction R which is associated with a cylindrical coordinate system centered on the tube axis. Let us suppose that the cross-section can be given uniquely at Pt by requiring that the sectioning plane have minimal sectional area. Let Q, the "hoop direction" at P t, be in the cross-sectional plane and perpendicular to R. Let Z, the "axial direction" at Pt, be perpendicular to R and Z, i.e. given by $Z = R \times Q$

Note that R, Q, and Z are, herein, always unit vectors – i.e. R is the "radial direction" rather than the radius vector. Similar to other curvilinear coordinate directions; R, Q, and Z are vector fields that are assumed to vary smoothly with position in the reference configuration.

As for the current configuration, let r, q, and z be defined via **R**, **Q**, and **Z** and the local deformation gradient tensor **F** as follows:

$$\mathbf{r} = \frac{\mathbf{F}^{-\mathrm{T}}\mathbf{R}}{|\mathbf{F}^{-\mathrm{T}}\mathbf{R}|}, \qquad \mathbf{q} = \frac{\mathbf{FQ}}{|\mathbf{FQ}|}, \qquad \mathbf{z} = \frac{(\mathbf{FQ}\cdot\mathbf{FQ})\mathbf{FZ} - (\mathbf{FQ}\cdot\mathbf{FZ})\mathbf{FQ}}{|(\mathbf{FQ}\cdot\mathbf{FQ})\mathbf{FZ} - (\mathbf{FQ}\cdot\mathbf{FZ})\mathbf{FQ}|}. \tag{2.1}$$

Assume our artery to undergo only deformations of uni-axial extension and inflation. Consider the artery then a straight, axis-symmetric tube undergoing an axis-symmetric deformation. As mentioned above the vectors R, Q, and Z correspond to the radial hoop, and axial directions for such a structure, and when an axis-symmetric deformation is imposed, r = R, q = Q, and z = Z. Due to symmetry, our stress tensor (**t**) does not depend on the cylindrical coordinates z and θ, whereby the equilibrium equations in the absence of body forces are:

$$\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}t\,(t_{rr} - t_{qq}) = 0, \qquad (8.1)_1$$
$$\frac{\partial t_{rq}}{\partial r} + \frac{2t_{rq}}{r} = 0, \qquad (8.1)_2$$
$$\frac{\partial t_{rz}}{\partial r} + \frac{2t_{rz}}{r} = 0. \qquad (8.1)_3$$

A direct consequence of these equilibrium equations is:

$$t_{rr}(r) = \int_{r_{(\mathrm{in})}}^{r} r^{-1}(t_{qq} - t_{rr})\,\mathrm{d}r + t_{rr}(r_{(\mathrm{in})}), \tag{8.2}$$

Where r (in) is the radius of the inner wall, and let r (out) be the radius of the outer wall. The internal pressure P(in)is −trr(r(in)) and the external pressure P(out) is−trr(r(out)). Central to the theory of hyper elasticity is the existence of materials that conserve, as strain energy, any mechanical work that is done on them. For mechanical processes that are isothermal and quasistatic with negligible body forces, conservation of energy and the stress power law, by substituting **t** for its respective strain energy components we yield the equation of balance for pressure in an axis symmetric tube:

$$P_{(\mathrm{in})} - P_{(\mathrm{out})} = \int_{r_{(\mathrm{in})}}^{r_{(\mathrm{out})}} \frac{2}{rJ}\frac{\partial \mathrm{W}}{\partial \gamma_3}\,\mathrm{d}r. \tag{8.3}$$

Hence, the transmural pressure difference depends on only one response function. Equation (8.2) is also useful for determining the load LW supported by the tube wall. Toward this end, multiply (8.2) by rdr and integrate trr(r) rdr from r(in)to r(out). Integration by parts, rearrangement and multiplication by 2π yields:

$$0 = P_{(\mathrm{in})}\pi r_{(\mathrm{in})}^2 - P_{(\mathrm{out})}\pi r_{(\mathrm{out})}^2 - \pi\int_{r_{(\mathrm{in})}}^{r_{(\mathrm{out})}} (t_{rr} + t_{qq})r\,\mathrm{d}r. \tag{8.4}$$

The integral of tzz 2π r dr from r(in)to r(out)is LW; however, when (8.4), i.e. zero,is added to this integral, it can be expressed as

$$L_W = \pi \int_{r_{(\text{in})}}^{r_{(\text{out})}} (2t_{zz} - t_{rr} - t_{qq}) r \, \mathrm{d}r + P_{(\text{in})} \pi r_{(\text{in})}^2 - P_{(\text{out})} \pi r_{(\text{out})}^2. \qquad (8.5)$$

By applying the stress power law and finding **t** in terms of its strain energy Lw simplifies as

$$L_W = 3\pi \int_{r_{(\text{in})}}^{r_{(\text{out})}} \frac{1}{J} \frac{\partial W}{\partial \gamma_2} r \, \mathrm{d}r + P_{(\text{in})} \pi r_{(\text{in})}^2 - P_{(\text{out})} \pi r_{(\text{out})}^2, \qquad (8.6)$$

Whereby the load depends on one response function.

Extension and Inflation of Incompressible, Homogeneous Tubes

The balance equations (Section above) are simple integrals of only one response function for incompressible and compressible tube materials, yet the experimental analysis is greatly simplified for incompressible materials because the deformation is entirely known from measurements of axial length and external diameter. Measurements (reference and current) of the axial length give the axial stretch λZ which far from edges should be constant throughout. Measurement of the external diameter multiplied by π gives the external circumference, and the current circumference divided by the reference circumference gives the hoop stretch on the exterior, λ Q(R(out)). The hoop stretch depends on radius, i.e. λQ(R); and since the volume of the wall remains constant, the hoop stretch at all radii is obtained in terms of our measured quantities by

$$\lambda_Q^2(R) = \lambda_Z^{-1} + \left(\lambda_Q^2(R_{(\text{out})}) - \lambda_Z^{-1}\right) \frac{R_{(\text{out})}^2}{R^2}. \qquad (9.1)$$

Our strain energy function defined by our previously chosen framework is definable in five terms since pure dilation is decoupled from distortional strains. Since our measurements were taken in terms of pure inflation and extension our strain energy function only depends on two terms γ^2 and γ^3 where they are defined as:

$$\gamma_2 = \ln \lambda_Z^{3/2}, \qquad \gamma_3(R) = \ln \lambda_Q^2(R) + \ln \lambda_Z \qquad (9.3)_{1-3}$$

With this in account our balance equations from above reduce to:

$$P = 2\int_{r_{(in)}}^{r_{(out)}} \frac{\partial W}{\partial \gamma_3}\frac{dr}{r}, \tag{9.4}$$

$$L_Z = 3\pi\int_{r_{(in)}}^{r_{(out)}} \frac{\partial W}{\partial \gamma_2} r\, dr. \tag{9.5}$$

To obtain a solution for the response functions in terms of the P andLZ, a change in variables is helpful. In particular, let W (γ2, γ3,0,0,0) = w(α, β), where:

$$\alpha = \exp(\gamma_2) - 1, \qquad \beta = \exp(\gamma_3) - 1. \tag{9.6}$$

In terms of our stretches α and β are:

$$\alpha = \lambda_Z^{3/2} - 1, \qquad \beta(R) = \lambda_Z\lambda_Q^2(R) - 1, \tag{9.7}$$

With this change in variables the balance equations can be expressed as

$$L_Z = 3\pi\lambda_Z^{1/2}\int_{R_{(in)}}^{R_{(out)}} \frac{\partial w}{\partial \alpha} R\, dR, \qquad P = 2\int_{R_{(in)}}^{R_{(out)}} \frac{\partial w}{\partial \beta}\frac{dR}{R}. \tag{9.9}$$

To do so, make the change of variables and use the following equations:

$$\begin{aligned} &\exp(\gamma_2) = \lambda_Z^{3/2}, \quad \exp(\gamma_3) = \lambda_Z\lambda_Q{}^2, \\ &R^{-1}r = \lambda_Q, \qquad r\, dr = \lambda_Z^{-1} R\, dR. \end{aligned} \tag{9.10}$$

The last one comes from the condition that each shell in the reference configuration (Z2πRdR) has the same volume in the current configuration (z2π r dr).Using (9.7) and (9.10), the incompressibility constraint in (9.1) can be expressed as:

$$\beta = \beta_{(out)}\left(\frac{R_{(out)}}{R}\right)^2. \tag{9.11}$$

It follows that powers of β are related to powers of β(out) via

$$\beta^i = (\beta_{(out)})^i\left(\frac{R_{(out)}}{R}\right)^{2i}. \tag{9.12}$$

This is particularly useful when we express ∂w/∂β, for example, as a power series of the form

$$\frac{\partial w}{\partial \beta} = \sum_{i=0}^{\infty}(\beta)^i B_i, \tag{9.13}$$

where the coefficients Bi are functions of α. Upon substitution of this series into $(9.9)_2$

$$P = 2\sum_{i=0}^{\infty}(\beta_{(\text{out})})^i B_i \int_{R_{(\text{in})}}^{R(\text{out})} \left(\frac{R_{(\text{out})}}{R}\right)^{2i} \frac{\mathrm{d}R}{R}. \tag{9.14}$$

This result is a power series of P in terms of β(out), and note that the integrals are easily evaluated. In particular,

$$P = \sum_{i=0}^{\infty}(\beta_{(\text{out})})^i P_i, \tag{9.15}$$

Where

$$P_i = \begin{cases} 2B_0 \ln\left(\dfrac{R_{(\text{out})}}{R_{(\text{in})}}\right), & i = 0, \\ \dfrac{B_i}{i}\left(\left(\dfrac{R_{(\text{out})}}{R_{(\text{in})}}\right)^{2i} - 1\right), & i \geqslant 0. \end{cases} \tag{9.16}$$

The above equation is easily inverted. It should now be evident that the power series in (9.13) and that in (9.15) are uniquely related for a given geometry of the reference configuration. For each set of coefficients *Bi* there is one set of coefficients Pi and vice versa. Hence, once we fit a suitable power series representation to the measured pressure in terms of β(out), then we may compute the corresponding, unique ∂w/∂β series and obtain ∂W/∂γ3 via (9.8).

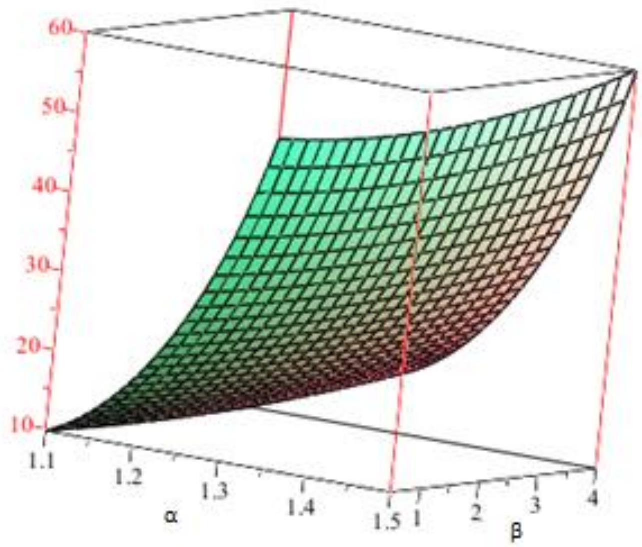
A likewise approach can be used to find the γ2 response function. For the two power series

$$\frac{\partial w}{\partial \alpha} = \sum_{i=0}^{\infty}(\beta)^i A_i, \qquad L_Z = \sum_{i=0}^{\infty}(\beta_{(\text{out})})^i L_i, \tag{9.17}$$

$(9.9)_1$ requires that the coefficients be uniquely related via

$$L_i = 3\pi\lambda_Z^{1/2} \begin{cases} \dfrac{A_0}{2}\left(R_{(\text{out})}^2 - R_{(\text{in})}^2\right), & i = 0, \\ A_1 R_{(\text{out})}^2 \ln\left(\dfrac{R_{(\text{out})}}{R_{(\text{in})}}\right), & i = 1, \\ \dfrac{A_i R_{(\text{out})}^2}{2i-2}\left(\left(\dfrac{R_{(\text{out})}}{R_{(\text{in})}}\right)^{2i-2} - 1\right), & i \geqslant 2. \end{cases} \tag{9.18}$$

From our inverse solution we generate the function W(α, β). To gather a more intuitive sense of our strain energy function β should be thought of as energy from straining in the radial direction or **R**. The α term is the energy stored from straining in axial direction. Thus our function W(α, β) can be seen as a measure of strain in the radial and axial directions.



Strain Energy: Alpha vs Beta vs Energy

Verification of correctness (and uniqueness) is easily performed by comparing our data to the inverse solution balance equations shown below the original data is transformed into terms of β and shown on the figure against pressure.

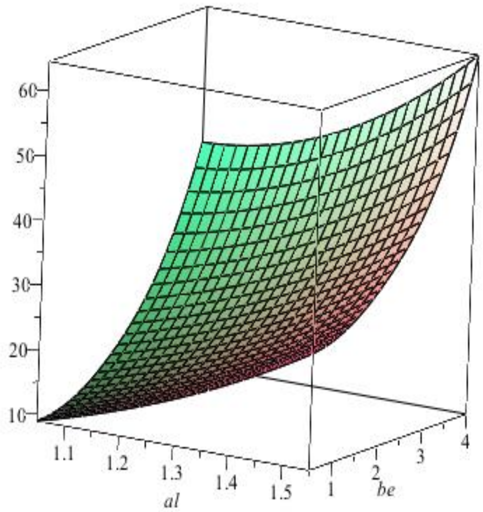
Results (See Appendix for condensation of all resulting figures)

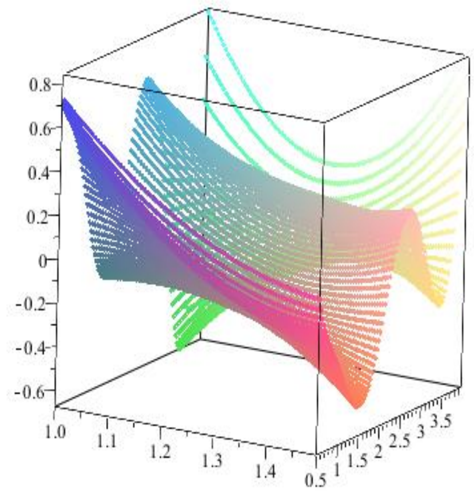
The data set used to obtain the following solution was taken from Takamizawa 1987 [11]. The following solution was obtained from analysis of the unique strain energy, pressure vs. diameter curves at fixed lengths, and load vs. diameter curves at fixed lengths.

$$W(\alpha, \beta) = \Gamma(\alpha) + \Gamma(\beta) + \Gamma(\beta) * \Omega(\alpha, \beta)$$

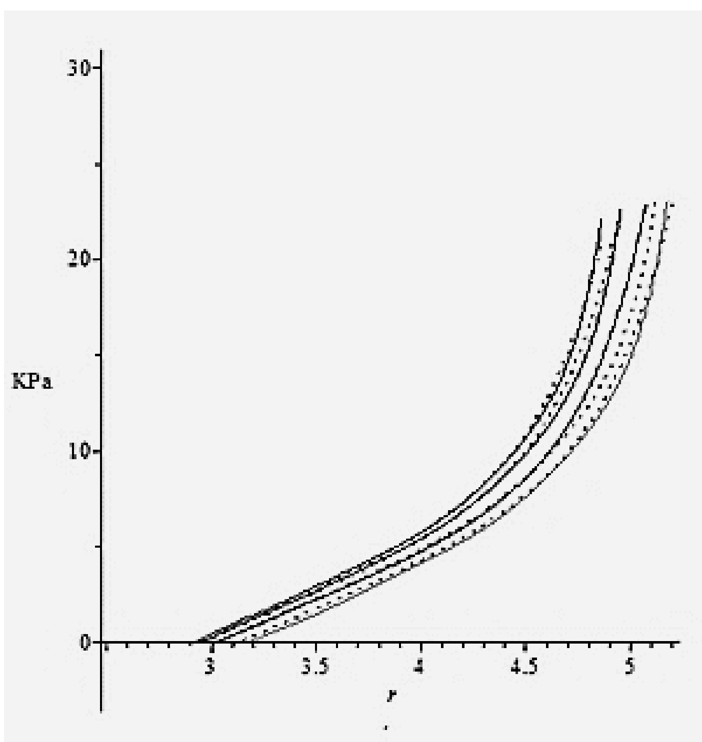
See appendix for detailed explanation of solution.

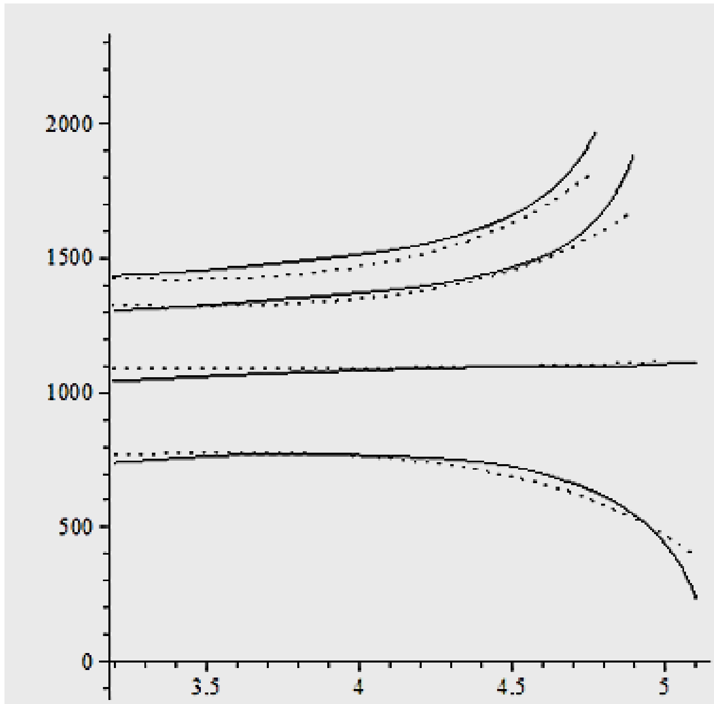
The Authors' strain energy function when a least squares fit was performed yielded the following surface and residual plot:





As prescribed in Criscione (2004) the strain energy function yielded the response functions of: 1) Pressure (Kpa) vs. Diameter (mm) 2) Load (N) vs. Diameter (mm):





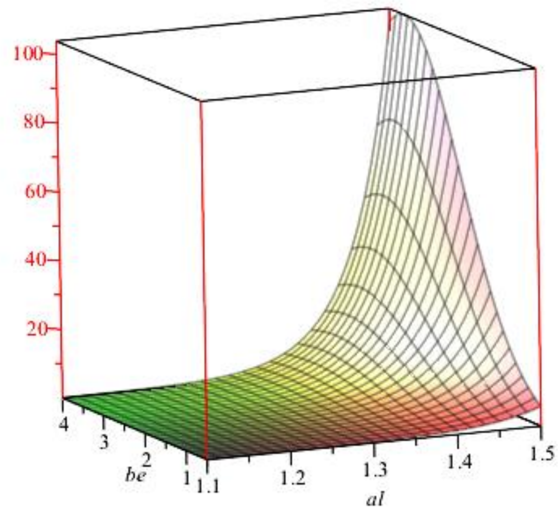
Discussion

It should be noted that this if this is the first display of strain energy for carotid arteries undergoing inflation and extension (which to the authors' knowledge it is) it is clearly by virtue of using the Criscione's framework which permits invertibility and minimizes covariance. By having the unique strain energy data rather than a statistically unlikely set as in previous attempts to model behavior and gain insight, the solution was able to be fine-tuned such that behavior was better captured. Future models/solutions/applications of vessels should take into account strain energy data as evidenced by the author's results.

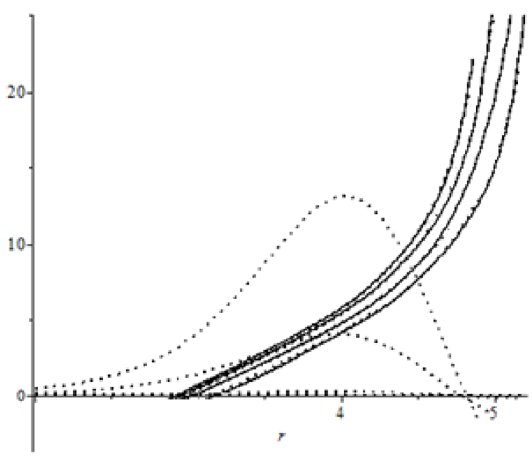
Other Model Fits to Strain Energy and resulting response functions:

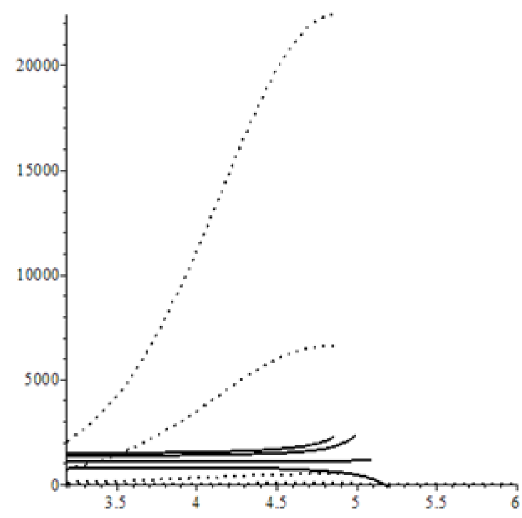
As was shown will be shown below, the inability to fit strain energy suggests that the model/solution/application does not capture the intrinsic behavior of a system.

For illustration, the authors took Fung's original model, used a least squares regression and the following resulted (following page):



From the above canonical strain energy the corresponding response functions for pressure and load resulted in:





The shape of the surface for the strain energy is an artifact of the framework used and organization of parameters. In the solution of the system there exists discrete "alpha" and "beta" terms and a cross term. Where as in Fung's strain energy (see reference [8]) all the parameters are grouped into one exponential term. This does not allow for enough flexibility to model the known strain energy. Which has three local maximum points which can be roughly characterized as: 1) (AlphaMax,BetaMin) 2) (AlphaMin,BetaMax) 3) (AlphaMax,BetaMax). If one views the solution, one could intuitively think about the points being captured by:

$$W(\alpha,\beta) = \Gamma(maxAlpha) + \sim 0 + \sim negligible$$

$$W(\alpha,\beta) = \sim 0 + \Gamma(maxBeta) + \sim negligible$$

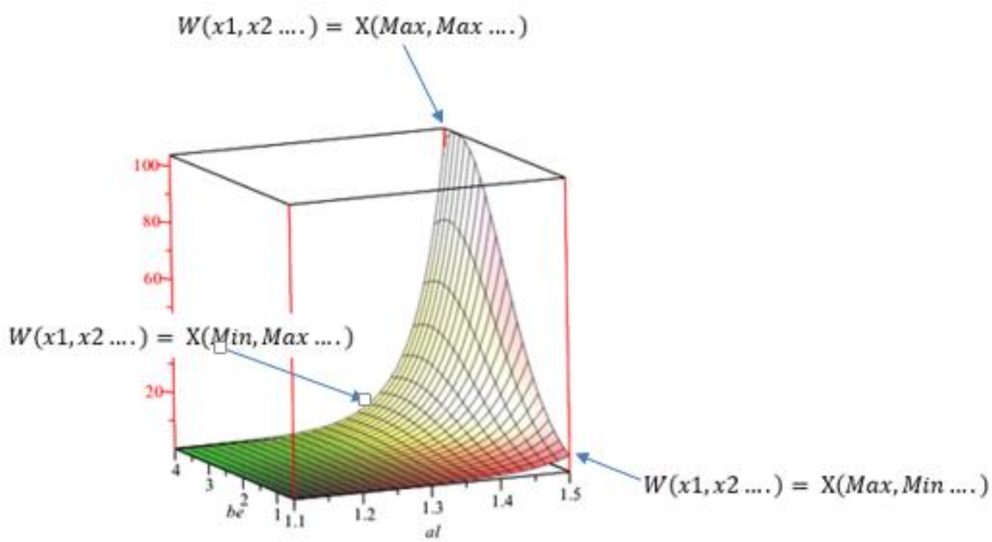
$$W(\alpha, \beta) = \Gamma(maxAlpha) + \Gamma(maxBeta) + \sim negligible$$

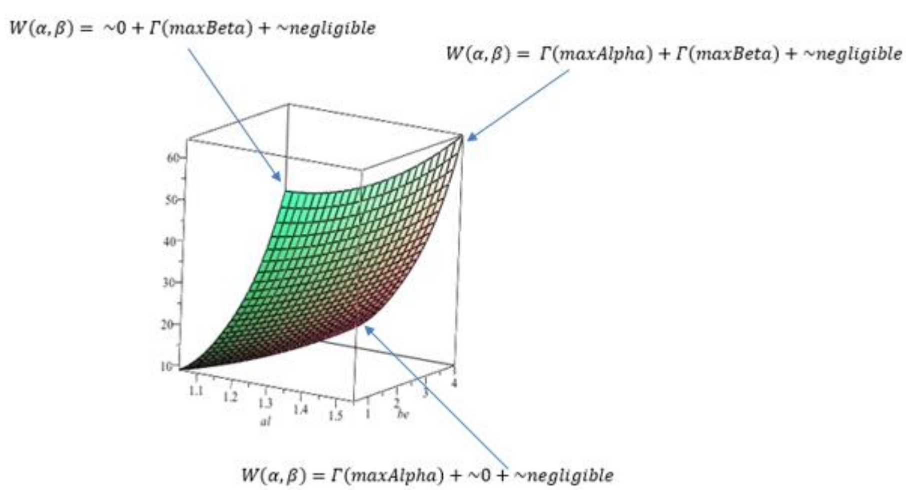
Which allows for capturing of the three central points where as previous models followed a canonical representation of:

$$W(x1, x2 \ldots.) = X(x1, x2 \ldots.)$$

The grouping of term as done in the canonical manner provides a large global maximum and relatively much smaller local maximums due to the nature of the function X generally being non-linear or exponential, which by definition causes great discrepancy in small perturbations.

The choice of grouping all parameters provides a global maximum but, too much discrepancy between the global maximum and local maximums.

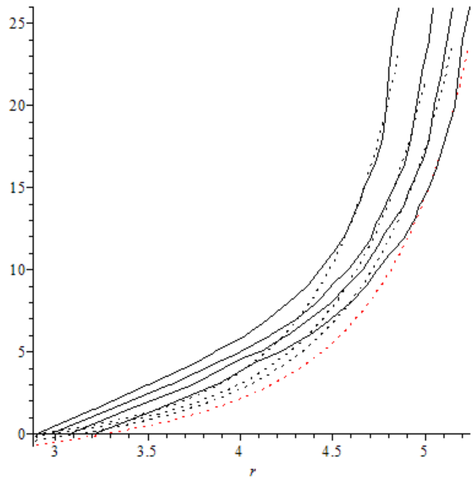


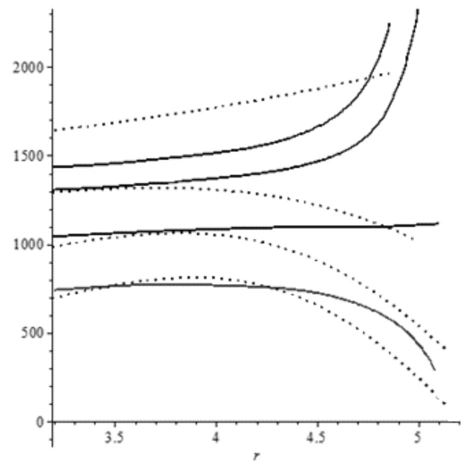


Whereas the solution leaves for less discrepancy by exploitation of non-covariant terms, which allows for isolated characterizations of alpha and beta terms.

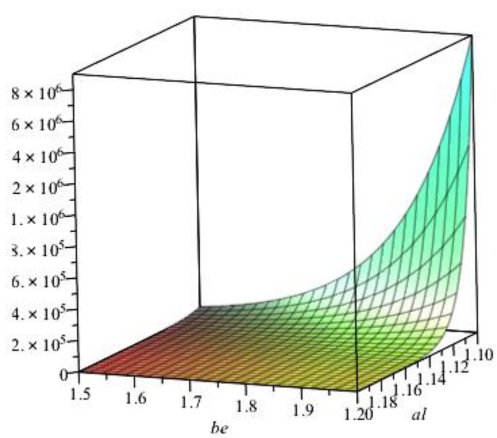
Other Model Fits to response function and resulting response functions:

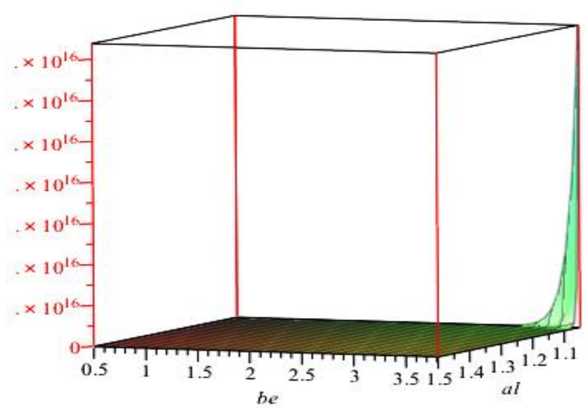
The authors’ aforementioned assertion was that it would be statistically unlikely that an over determined models strain energy would match that of the systems inherent strain energy.



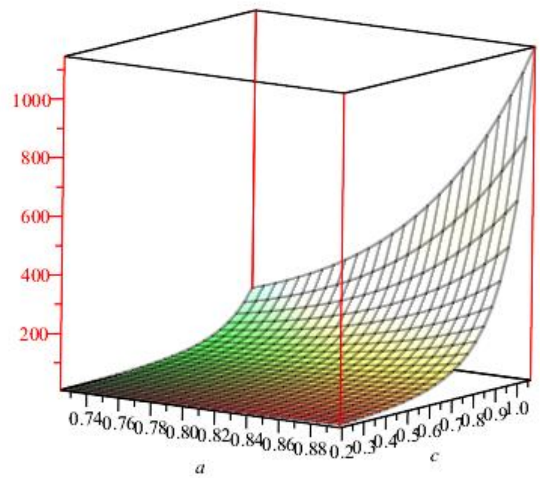


The resulting strain energy is both figures below, for clarity the axes were chosen to better see the shape for smaller perturbations from the origin.





The authors' would like to note that during our initial investigation we examined Fung's strain energy from his parameters. And it is in the authors' opinion is remarkably close in comparison to the provided strain energy shown above:



Evidence to why the solution is practical

The authors claim evidence rather than proof because of the lack of formal structure in this area of biomechanics. Without at least a set of axioms that are a part of a general consensus on what a 'solution' is we can only claim our solution to be practical.

For an anecdotal treatment consider an analogue pressure gauge, now consider an observer who can differentiate all positions theta (the angle the analogue pressure gauge makes with device). The authors' then pose how many angle's theta exist such that the pressure gauge to an average observer (You or I) exist such that the gauge could read "1Kpa?" The answer is *uncountbly infinite*.

With this in mind we now examine the proposed solution. Since there exist uncountlbly infinite solutions what are we defining to be a solution for our posed problem? But, therein lies the true question what is our problem? If we are trying to model strain energy as found in Takamizawa, well the solution matches orders of magnitude greater than that of a previous attempts.

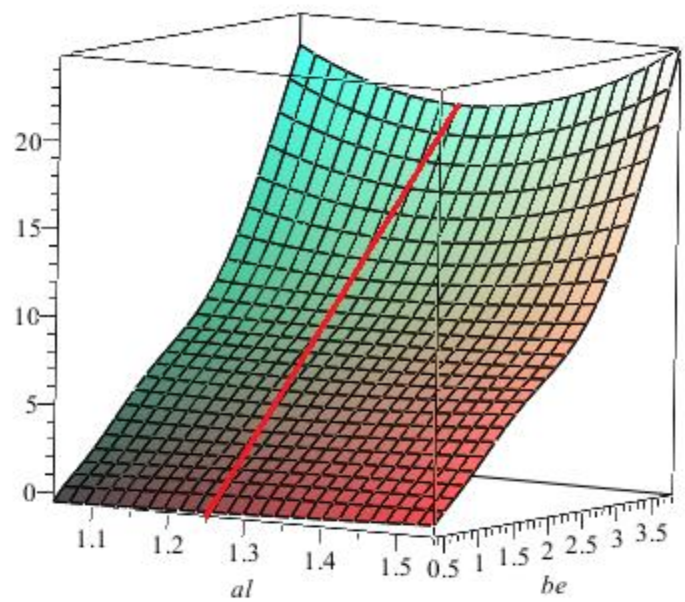
But to the keen reader the usage of the word of solution was more than likely a prick in their side. A solution would be the "exact fit". Which our proposed solution does not "exactly fit", if one looks at our residual plot we only have an exact fit for the oscillatory points which intersect the plane.

Then why can we call our proposed model a solution? We assert that since our model is reasonably close and within general experimental error (+-5%) in all regimes then rather than a model we have a solution, whereas all of the "models" proposed in this paper and to the knowledge of the author do not meet this criterion. This is still a positive definite representation of a natural system. Which in reality never occurs.

The authors' present two pieces of evidence which we claim point to our solution being one of uncountable infinite solutions that fit the data. We assert our solution is practical since it gives practical insight into cardiovascular physiology and our conjecture that our multidimensional construct still behaves neo-hookean from a collapsed dimensional perspective. Secondly, when our solution is compared to an orthogonal solution all behavior which is not purely alpha, or purely beta is captured in the $\Gamma(\beta) * \Omega(\alpha, \beta)$ or non-pure term (cross term for a more canonical description).

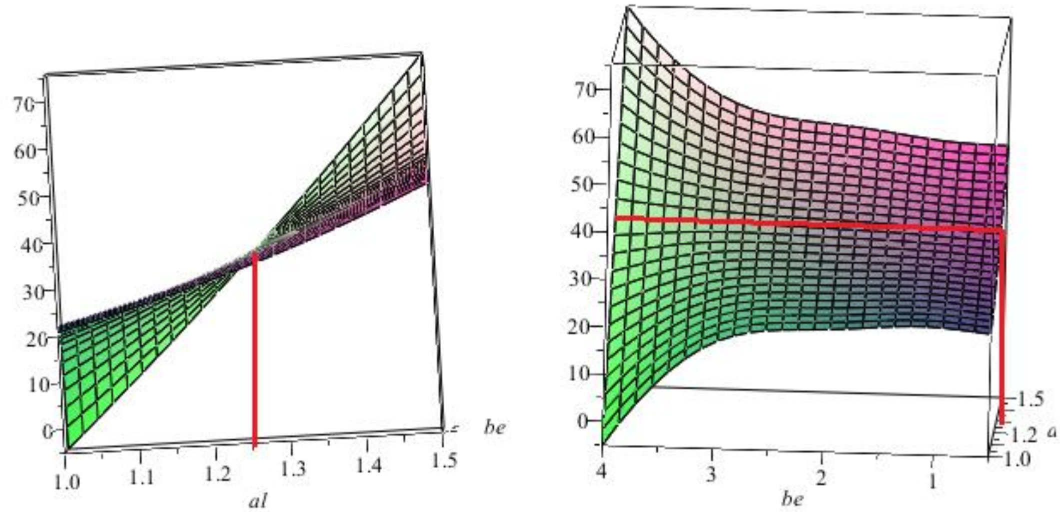
Proof of conjecture

In the authors' early stages of analysis we conjectured that there existed an arterial stretch such that the artery required the least amount of energy to operate. Examination of the strain energy derivatives with respect to alpha and beta give evidence supporting the assertion, the figures are shown below:



d(W)/dβ

The red line shown above indicates the local minimum for the maximum curve. If this point is examined in the derivative with respect to alpha we see saddle point behavior. (Following page)



dW/dα

In order to claim 'proof' of our conjecture, we have to have irrefutable evidence according to the general consensus of mathematician's through history. Thus, we assert, if the indicated local minimum/saddle point indicated in the derivatives arose and **was not** a local nor global minimum in the strain energy, we have rigorously procured a physiologic stretch that produces a minimal conformation.

In our Ω(α,β) we have a term that is expressed as $(\alpha - c)^2$ where c is a functional parameter. If there existed a stretch length α that truly was the 'energy minimum' it would arise here due to the functional nature of the solution. Our conjecture was this value should be 1.25 as the red lines above indicate. The following was our result:

$$W(\textit{Regression fixing } 1.25)\, 11.58\,\alpha^2 - \frac{15.30\cosh(0.08\,\alpha)}{\alpha^2+1} + 1.56\,\beta^2 + \frac{2.58\cosh(1.50\,\beta)}{1+\beta^2} + \frac{20.6\cosh(1.16\,\beta)\left((\alpha-1.25)^2 - 0.08\,\beta\right)}{1+\beta^2}:$$

$$W(\textit{Regression } \mathbf{for} \text{ } 1.25 \textit{ position}) := 11.35\,\alpha^2 - \frac{14.68\cosh(0.08\,\alpha)}{\alpha^2+1} + 1.54\,\beta^2 + \frac{2.73\cosh(1.50\,\beta)}{1+\beta^2} + \frac{19.51\cosh(1.18\,\beta)\left((\alpha-1.245)^2 - 0.08\,\beta\right)}{1+be^2}:$$

The authors' assert our solution has granted some insight into the mechanical properties of carotid arteries thus making it a 'practical' solution.

Cross Term Behavior is captured in $\Gamma(\beta) * \Omega(\alpha, \beta)$ *term*

The canonical model for an orthogonal system whose parts both have individual contributions and synergistic contributions (non-linear effects that arise from the individuals acting as conglomerate) is as follows:

$$F(x, y) = f(x) + g(y) + h(x, y)$$

Where one could reasonable conclude that *f* captures the individual contributions of the x variable, *g* the y variable and *h* the synergistic behavior. It can be easily shown that from the inverse solution- (a polynomial of 9th order with 27 parameters) which produced the initial strain energy can be decomposed into alpha beta and synergistic contributions, and subsequently when the proposed solution is decomposed into its respective parts (alpha, beta, synergistic) the individual contributions match in shape and magnitude, thus evidencing the solution capturing isolated and cross behavior, further inferring the practicality of the solution.

Conclusion

A three-dimensional stress-strain relationship is determined from a strain energy function of an original type for the arterial wall from data obtained under inflation and longitudinal stretch experiments. Since the results produced by the strain energy function are a match under experimental conditions, we consider the function one of unaccountably many solutions that can accurately represent the data. We call the solution practical since one can gain a better understanding about circulatory physiology from its construction. The purpose of this paper was to present the necessity of the usage of a constitutive frame work that permits invertitibility and to show that models that do not fit in responses functions are unlikely to fit in strain energy and vice versa. Further work should be done into including a torsional term to have a more complete solution for naturally behaving carotid arteries.

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12. It will lastly be noted I finished writing this paper exactly at 4:20 p.m. of 4/8/14 and started 5/4/2011… it was not rushed