

A Full Geometric Realization of S^2 in Two Dimensions

Abstract

We construct the two-sphere S^2 entirely within two dimensions. Starting from its stereographic metric on \mathbb{R}^2 ,

$$g_s = \frac{4}{(1+r^2)^2} (dr^2 + r^2 d\theta^2),$$

the space is incomplete at the north pole. Metric completion adds a single limit point, and in reciprocal coordinates $\rho = 1/r$, the metric extends smoothly across $\rho = 0$, confirming a genuine 2-manifold structure. The result is a compact, smooth, and complete surface—an intrinsic realization of S^2 within two-space itself.

1 Introduction

It is standard to describe the two-sphere S^2 as a surface embedded in \mathbb{R}^3 , equipped with the round metric induced from the ambient space. Under stereographic projection from the north pole, $S^2 \setminus \{N\}$ identifies with \mathbb{R}^2 , and the round metric pulls back to

$$g_s = \frac{4}{(1+r^2)^2} (dr^2 + r^2 d\theta^2).$$

In this conventional construction, the north pole does not appear in the coordinate chart, so the planar representation describes an *open patch* of the sphere. Here we treat (\mathbb{R}^2, g_s) not as a local chart, but as an *autonomous metric space*. We show that its *metric completion* naturally yields a smooth, compact, two-dimensional manifold isometric to the standard sphere. Hence, the geometry of S^2 arises entirely within two-space, without any reference to \mathbb{R}^3 .

2 The Spherical Metric on \mathbb{R}^2

Let (r, θ) denote polar coordinates on \mathbb{R}^2 . The metric

$$g_s = \frac{4}{(1+r^2)^2} (dr^2 + r^2 d\theta^2)$$

is the stereographic pullback of the round metric and has constant positive curvature $+1$. The radial geodesic distance from the origin to a point of radius r is

$$d_{g_s}(0, r) = \int_0^r \frac{2 dr'}{1+(r')^2} = 2 \arctan(r).$$

As $r \rightarrow \infty$, this tends to π , so (\mathbb{R}^2, g_s) is *incomplete*: a finite-distance limit point is missing.

3 The Added Point and Smooth Extension

We identify the missing point via completion and verify smoothness at that point.

3.1 Spherical Metric and Stereographic Projection

The round metric on S^2 in spherical coordinates (ψ, θ) is

$$ds^2 = d\psi^2 + \sin^2(\psi) d\theta^2,$$

where ψ is the polar angle from the south pole and θ the azimuthal angle. Under stereographic projection from the north pole onto the plane $z = 0$, the planar coordinates (x, y) satisfy

$$x = 2 \cos \theta \tan\left(\frac{\psi}{2}\right), \quad y = 2 \sin \theta \tan\left(\frac{\psi}{2}\right),$$

so the radial coordinate is

$$r = \sqrt{x^2 + y^2} = 2 \tan\left(\frac{\psi}{2}\right).$$

Inverting, $\tan(\psi/2) = r/2$.

3.2 Pullback of the Metric to the Plane

Differentiating $r = 2 \tan(\psi/2)$ gives

$$dr = \sec^2(\psi/2) d\psi = \left(1 + \frac{r^2}{4}\right) d\psi.$$

Using the half-angle identity $\sin(\psi) = \frac{2 \tan(\psi/2)}{1 + \tan^2(\psi/2)} = \frac{4r}{r^2 + 4}$, a straightforward substitution into $ds^2 = d\psi^2 + \sin^2(\psi) d\theta^2$ yields

$$g_s = \frac{4}{(1 + r^2)^2} (dr^2 + r^2 d\theta^2),$$

the standard stereographic formula.

3.3 Behavior Near the North Pole and Smoothness

As $\psi \rightarrow \pi$ (the north pole), we have $r \rightarrow \infty$. Introduce reciprocal coordinates $\rho = 1/r$; then $\rho \rightarrow 0$ corresponds to the added point. Compute

$$dr = -\frac{d\rho}{\rho^2}, \quad dr^2 = \frac{d\rho^2}{\rho^4}, \quad r^2 d\theta^2 = \frac{1}{\rho^2} d\theta^2,$$

and note

$$\frac{4}{(1 + r^2)^2} = \frac{4}{\left(1 + \frac{1}{\rho^2}\right)^2} = \frac{4\rho^4}{(1 + \rho^2)^2}.$$

Therefore,

$$g_s = \frac{4\rho^4}{(1 + \rho^2)^2} \left(\frac{d\rho^2}{\rho^4} + \frac{1}{\rho^2} d\theta^2 \right) = \frac{4}{(1 + \rho^2)^2} (d\rho^2 + \rho^2 d\theta^2).$$

This tensor is C^∞ and positive-definite at $\rho = 0$, so g_s extends smoothly across the added point. Inverting back to (ψ, θ) recovers the spherical metric

$$g_s = d\psi^2 + \sin^2(\psi) d\theta^2,$$

confirming smoothness at the completion point.

Proposition 1 (Intrinsic north pole and completion). *Let (\mathbb{R}^2, g_s) be as above. Then:*

1. *Any two sequences (r_n, θ_n) and (r_n, θ'_n) with $r_n \rightarrow \infty$ are at g_s -distance tending to 0.*
2. *The metric completion adds a single limit point $\{\infty\}$, represented by the metric limit $\lim_{r \rightarrow \infty} (r \cos \theta, r \sin \theta)$.*
3. *The completed space $(\overline{\mathbb{R}^2}, g_s)$ is a smooth, compact, complete 2-manifold isometric to the round sphere (S^2, g_{round}) .*

4 The Completed Space

Denote the completion by

$$\mathbb{R}_+^2 := \mathbb{R}^2 \cup \{\infty\}.$$

Equipped with g_s , the completed space (\mathbb{R}_+^2, g_s) is smooth, compact, and complete, and it is isometric to the round sphere (S^2, g_{round}) . Topologically, \mathbb{R}_+^2 is a one-point compactification of the plane; geometrically, it is the *full two-sphere*, expressed entirely in two dimensions.

5 Conclusion

The completed spherical plane

$$\mathbb{R}_+^2 = \overline{(\mathbb{R}^2, g_s)}$$

provides a *full, intrinsic realization of S^2* in two dimensions. It remains locally two-dimensional everywhere, yet, it is not the Euclidean plane.

References

- [1] do Carmo, M. P. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.
- [2] Lee, J. M. *Introduction to Smooth Manifolds*. Springer, 2012.
- [3] Spivak, M. *A Comprehensive Introduction to Differential Geometry, Vol. 1–5*. Publish or Perish, 1979.
- [4] Gallot, S., Hulin, D., and Lafontaine, J. *Riemannian Geometry*. Springer, 2004.
- [5] Kühnel, W. *Differential Geometry: Curves – Surfaces – Manifolds*. AMS, 2015.
- [6] Nash, J. "The Imbedding Problem for Riemannian Manifolds." *Annals of Mathematics*, 63(1):20–63, 1956.
- [7] Munkres, J. *Topology*. Prentice-Hall, 2000.

- [8] Boothby, W. M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, 2003.
- [9] O'Neill, B. *Elementary Differential Geometry*. Academic Press, 2006.
- [10] Burke, W. L. *Applied Differential Geometry*. Cambridge University Press, 1985.
- [11] Willmore, T. J. *An Introduction to Differential Geometry*. Oxford University Press, 1993.
- [12] Tu, L. W. *An Introduction to Manifolds*. Springer, 2010.
- [13] Ahlfors, L. V. *Complex Analysis*. McGraw-Hill, 1979.
- [14] Hicks, N. J. *Notes on Differential Geometry*. Van Nostrand, 1965.
- [15] Kobayashi, S. and Nomizu, K. *Foundations of Differential Geometry, Vol. 1*. Wiley-Interscience, 1996.